# Assignment 2.

This homework is due *Thursday* Feb 2.

There are total 37 points in this assignment. 33 points is considered 100%. If you go over 33 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

(1) [3pt] (2.2.1) Show that if $a, b \in \mathbb{Z}$, with $b > 0$, then there exist unique integers $q$ and $r$ satisfying $a = qb + r$, where $2b \le r < 3b$.

(2) [2pt] (2.2.2) Show that any integer of the form $6k + 5$ is also of the form $3j + 2$, but not conversely.

(3) [2pt] (2.2.6) Show that cube of any integer is of the form $7k$ or $7k \pm 1$.

(4) [3pt] Show that in every Pythagorean triple, at least one number is divisible by 3. (Pythagorean triple is three integers $a, b, c$ such that $a^2 + b^2 = c^2$.)

(5) [2pt] Express $2012_3$ in binary.

(6) [4pt] Prove that every number of the form $111\ldots11_9$ is triangular.

(7) (a) [5pt] Explain the following algorithm to multiply two positive integers (i.e. explain why it always works).
To multiply two numbers, form two columns with these numbers in the top entries. For each new row, halve the number in the left column, always rounding down, and double the number in the right column. When the left number reaches 1, strike out all rows where the left number is even. To get the answer, sum up the remaining numbers in the right column. For example, to multiply $75 \times 221$, write

| | |
|---|---|
| 75 | 221 |
| 37 | 442 |
| ~~18~~ | ~~884~~ |
| 9 | 1768 |
| ~~4~~ | ~~3536~~ |
| ~~2~~ | ~~7072~~ |
| 1 | 14144 |
| | 16575 |

Comment. This algorithm is called Peasant Multiplication and was used in a number of ancient cultures when "normal" long multiplication was not accessible. For that there could be two reasons: (1) either positional system was not adopted by the culture; or (2) those who were doing the multiplication lacked education to know the multiplication table — hence the name of this algorithm.

(b) [4pt] Will the same work if we divide and multiply by 3 instead of halving and doubling? If yes, explain why; if not, give an example of two numbers for which it fails and suggest an adjustment to the algorithm that would make it work.

(8) (a) [5pt] Given a single pan scale, find the least number of weights and their values necessary to in order to weigh all integral weights from 1 pound to 50 pounds.
(b) [4pt] Same question, except you are asked about number of *different* weights necessary if you are allowed to use up to 2 of each weight.

(9) [3pt] For every $n \in \mathbb{Z}$, $n > 0$, show that $1 + 2 + \ldots + n$ divides $3(1^2 + 2^2 + \ldots + n^2)$.